Extract

Mathematics Project Competition for Secondary Schools (2020/21)

Category A (Junior secondary project)

Title: Don “丼” inside a quadrilateral

Table of Content

1. Introduction

In this paper, we are going to investigate the ratio of inner area and outer area of a quadrilateral under certain conditions. First, we start with a simple case by considering a rectangle. In figure 1, *ABCD* is a rectangle with $AE = EF = FB$, $BG = GH = HC$, $CI = IJ = JD$ and $DK = KL = LA$, it is obvious that we have

$$\text{area of } GHKL = \frac{1}{3} \times \text{area of } ABCD,$$

$$\text{area of } MNOP = \frac{1}{9} \times \text{area of } ABCD,$$

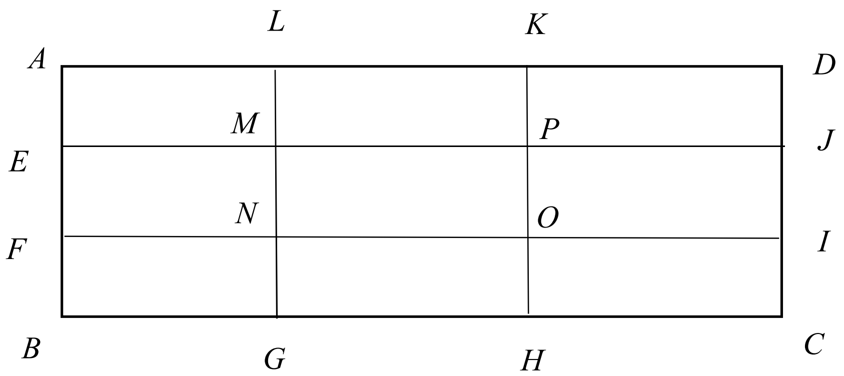

Figure 1

Next, we would like to investigate a more general condition by considering different ratio of the sides. In figure 2, *ABCD* is a rectangle with $AE : EF : FB = DJ : JI : IC = r : s : r$ and $BG : GH : HC = AL : LK : KD = p : q : p$. By letting $EF = sk$ and $GH = ql$, where *k* and *l* are non-zero constants, we can easily deduce that

$$\text{area of } GHKL = \frac{q}{2p+q} \times \text{area of } ABCD,$$

$$\text{area of } MNOP = \frac{qs}{(2p+q)(2r+s)} \times \text{area of } ABCD,$$

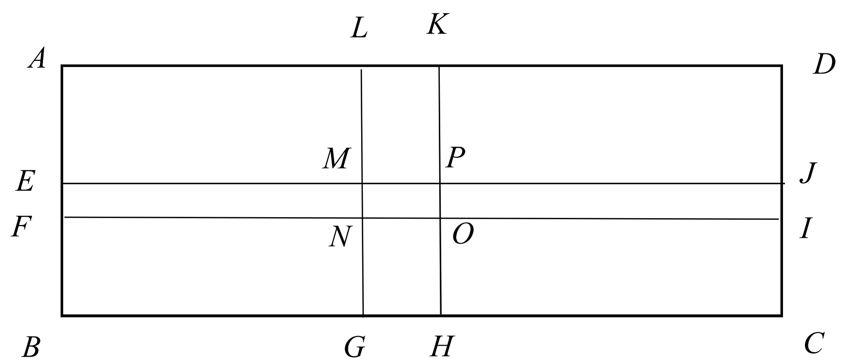

Figure 2

Now, we are interested in asking the following questions: "if $ABCD$ is considered as a parallelogram, a trapezium or a convex quadrilateral, can the result still hold?" and "if the result holds, how do we prove the result?".

Here we are going to investigate the case of parallelogram. In figure 3, $ABCD$ is a parallelogram with $AE = EF = FB$, $BG = GH = HC$, $CI = IJ = JD$ and $DK = KL = LA$, since $AD \parallel BC$ and $AL = BG$, we have $ABGL$ is a parallelogram, which implies $AB \parallel LG$. Similarly, we can deduce that $AD \parallel EJ \parallel FI \parallel BC$ and $AB \parallel LG \parallel KH \parallel DC$. So, we have $AL = LK = KD = EM = MP = PJ = FN = NO = OI = BG = GH = HC$ and $AE = EF = FB = LM = MN = NG = KP = PO = OH = DJ = JI = IC$. Therefore, we have

$$\text{area of } GHKL = \frac{1}{3} \times \text{area of } ABCD,$$

$$\text{area of } MNOP = \frac{1}{9} \times \text{area of } ABCD.$$

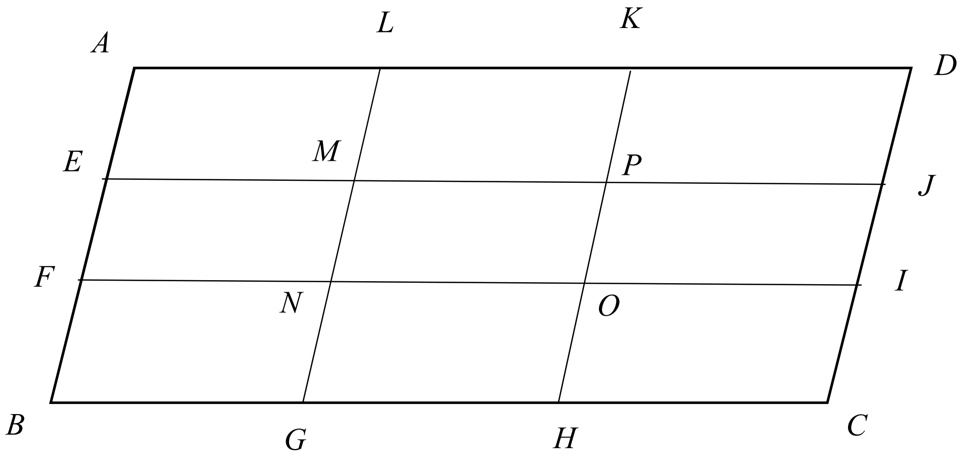

Figure 3

Next, in figure 4, $ABCD$ is a parallelogram with $AE : EF : FB = DJ : JI : IC = r : s : r$ and $BG : GH : HC = AL : LK : KD = p : q : p$. Similar to the previous reasoning, we can deduce that

$$\text{area of } GHKL = \frac{q}{2p+q} \times \text{area of } ABCD,$$

$$\text{area of } MNOP = \frac{qs}{(2p+q)(2r+s)} \times \text{area of } ABCD,$$

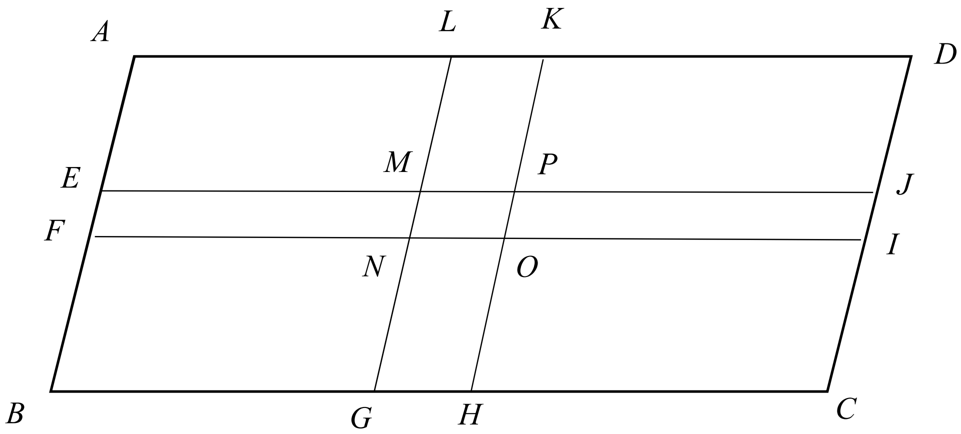

Figure 4

Now, we are going to investigate the case in a trapezium. In figure 5, *ABCD* is a trapezium with *AD* // *BC*, $AE = EF = FB$, $BG = GH = HC$, $CI = IJ = JD$ and $DK = KL = LA$, since *AD* // *BC*, we have the height of *GHKL* and the height of *ABCD* are the same. Also, *GHKL* and *ABCD* are trapeziums, by using the formula of area of a trapezium (i.e. area of a trapezium = $\frac{1}{2}\times$ (upper base + lower base) $\times$ height), we can deduce that

$$\text{area of } GHKL = \frac{1}{3} \times \text{area of } ABCD.$$

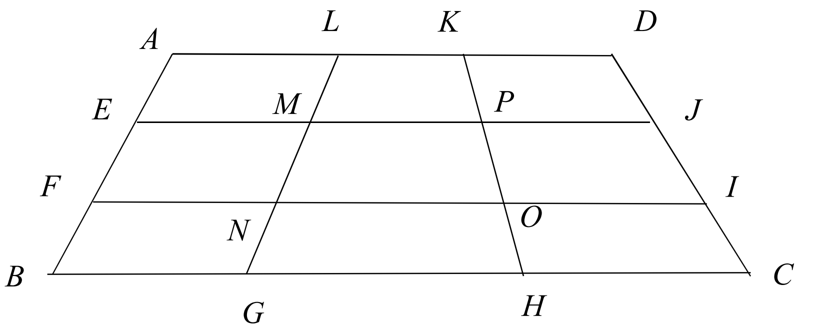

Figure 5

However, when we are going to consider the ratio of the area of *MNOP* to the area of *ABCD*, we need more detailed work. We are going to show the detailed work of proof in a convex quadrilateral later.

4. Conclusions and Suggestions

In this paper, we are inspired by the ratio of a part of inner area to the outer area in a rectangle and parallelogram, so we tried to investigate the ratio of a part of inner area to the outer area in a convex quadrilateral with different ratios on the sides. We have achieved that there are some formulas between the inner area and the outer area of a convex quadrilateral.

We are still interested in studying the ratio of the inner area to the outer area at the corners of a quadrilateral. For example, when we impose some certain conditions on a quadrilateral, we can deduce some interesting results. The following is one of the interesting results under a certain condition.

In figure 13, *ABCD* is a trapezium with *AD // BC, AD* : *BC* =1:*r* , *AP* = *PQ* = *QB*, *BH* = *HG* = *GC*, *CR* = *RS* = *SD*, *AE* = *EF* = *FD*. Assume the area of *ABCD* =*w*.

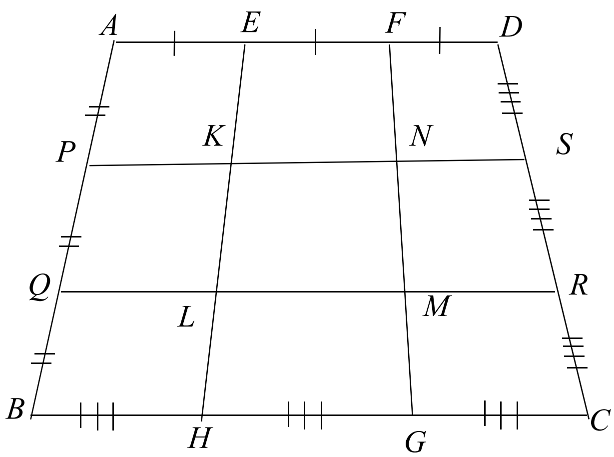

Figure 13

By apply lemma, we have $PK = KN = NS$, $QL = LM = MR$, $EK = KL = LH$ and $FN = NM = MG$. Besides, we can prove that *AD* // *PS* // *QR* // *BC* by the following reasons.

Construct a point S' on the line segment *CD* such that PS' // *AD* // *BC*, since $AP : PB = 1:2$ and PS' // *AD* // *BC*, we have $DS' : S'C = 1:2$ due to "the generalized intercept theorem". As *DS* :*SC* =1:2, $S' = S$. Therefore, we have *PS* // *AD* // *BC*. Similarly, we can also deduce that *QR* // *AD* // *BC*. So, we have *AD* // *PS* // *QR* // *BC*.

By the formula of area of a trapezium (i.e. area of a trapezium = $\frac{1}{2}\times$ (upper base + lower base) $\times$ height),

we can obtain area of *BHEA*= area of *HGFE* = area of *GCDF* = $\frac{w}{3}$.

Note that: area of ΔAEB : area of $\Delta BEH = AE : BH = 1 : r$.

$\therefore$ area of $\Delta AEB = \frac{1}{1+r}\times\frac{w}{3}=\frac{w}{3(1+r)}$ and area of $\Delta BEH = \frac{r}{1+r}\times\frac{w}{3}=\frac{wr}{3(1+r)}$

area of $PKEA$

= area of ΔAPE + area of ΔPKE

$= \frac{1}{3}\times$area of ΔAEB + $\frac{1}{3}\times$area of ΔPHE

$=\frac{w}{9(1+r)}+\frac{1}{3}(\frac{w}{3}-$area of $\Delta PAE -$ area of $\Delta PBH)$

$=\frac{w}{9(1+r)}+\frac{1}{3}[\frac{w}{3}-\frac{1}{3}(\frac{w}{3}-$area of $\Delta BEH)-\frac{2}{3}(\frac{w}{3}-$area of $\Delta AEH)]$

$=\frac{w}{9(1+r)}+\frac{1}{3}(\frac{1}{3}\times$area of $\Delta BEH+\frac{2}{3}\times$area of $\Delta AEH)$

$$=\frac{w}{9(1+r)}+\frac{1}{3}(\frac{1}{3}\times\frac{wr}{3(1+r)}+\frac{2}{3}\times\frac{w}{3(1+r)})$$

$$=\frac{w}{9(1+r)}+\frac{1}{3}(\frac{wr}{9(1+r)}+\frac{2w}{9(1+r)})$$

$$=\frac{(5+r)w}{27(1+r)}$$

Then the area of $BHLQ=\frac{w}{3}-\frac{w}{9}-\frac{(5+r)w}{27(1+r)}=\frac{w(1+5r)}{27(1+r)}$

Form the above, it is noted that

area of $AEKP$ = area of $EFNK$ = area of $FDSN = \frac{(5+r)w}{27(1+r)}$

area of $PKLQ$ = area of $KNML$ = area of $NSRM = \frac{w}{9}$

area of $QLHB$ = area of $LMGH$ = area of $MRCG = \frac{w(1+5r)}{27(1+r)}$.

It is interesting to explore more special cases about the ratio of the inner area to the outer area in a quadrilateral. However, due to the limited time, we could only achieve some result in certain conditions. We hope we can explore more properties in the future.

5. Students' Reflections

Student 1

It is interesting and meaningful to participate in this project. I learnt a lot from this investigation and also learnt how to work with my teammates. I am also grateful to my teammates and teachers. This project cannot be finished without their help and support.

Student 2

I am appreciated that my teachers, Mr. Chan and Mr. Tsang always support our thought. Also, I am thankful to my team members as this project cannot be finished without their hard work and assistance. In the end, I am grateful to my school, I will not be able to participate in this project without my school's support.

Student 3

By joining this project, I have learnt how to solve mathematics problems. Also, I am thankful to my teammates who have been helping me when I face the problems during working on the project. I felt appreciated to participate in the project and want to say thank you to our teachers who are always supporting us. Finally, I learn teamwork is very important during the investigation.

Student 4

During the days I spent on this project, I learnt a lot more about doing proofs and how to solve a problem starting from scratch. When I was studying different problems, my interest in mathematics was boosted. I explored my interest in doing research-like or investigation tasks. I am very grateful to be part of this project and have great teammates to finish the project together. Of course, I must say thank you to our teachers for their encouragement and support.

Student 5

In this math project, I learnt how to solve a problem with little piece of information. We need to try all the possible ways to solve the problems. Then, we drew the figures and explained our ideas step by step using the knowledge I have learned. Through this project, I was led to try to generalize a result from a particular case. It is really interesting and challenging ! All that knowledge I have gained in this project is what I cannot learn from math textbooks! At the same time, I also learned how to cooperate with my teammate and I really want to say thank you to our teachers. They encouraged us a lot!

6. References

1. 張景中 : “面積關係幫你解題”，商務印書館(香港)有限公司

2. 宋釗宜 : ”數學謎題入門”，大展出版社有限公司