Celebrating Marion's Theorem:

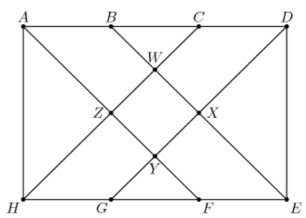
Correlations, Construction and Connections

1. Introduction

The large range of variations in triangles has always been fascinating to us. While researching for this competition, we found out about Marion's Theorem. This theorem is typically proven with analytical approaches, but in this case, an attempt at proving the theorem using a deductive approach is made. It inspired us to prove this theorem by only using the properties of triangles and shapes that we have learnt. We also decided to challenge ourselves and expand on her discovery.

Furthermore, while preparing for the American Mathematical Competition this year, we came across a problem on trisecting line segments.[1] It inspired us to extend Marion's Theorem and apply it to quadrilaterals.

In rectangle $ADEH$, points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$. In addition, $AH = AC = 2$, and $AD = 3$. What is the area of quadrilateral $WXYZ$ shown in the figure?



(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{3}}{2}$ **(D)** $\sqrt{2}$ **(E)** $\frac{2\sqrt{3}}{3}$

[1] *Art of Problem Solving. (2006). AMC 10A Problems/Problem 17. Retrieved from https://artofproblemsolving.com/wiki/index.php/2006_AMC_10A_Problems/Problem_17 Problem 17, 2006 American Mathematical Competition (10A)*

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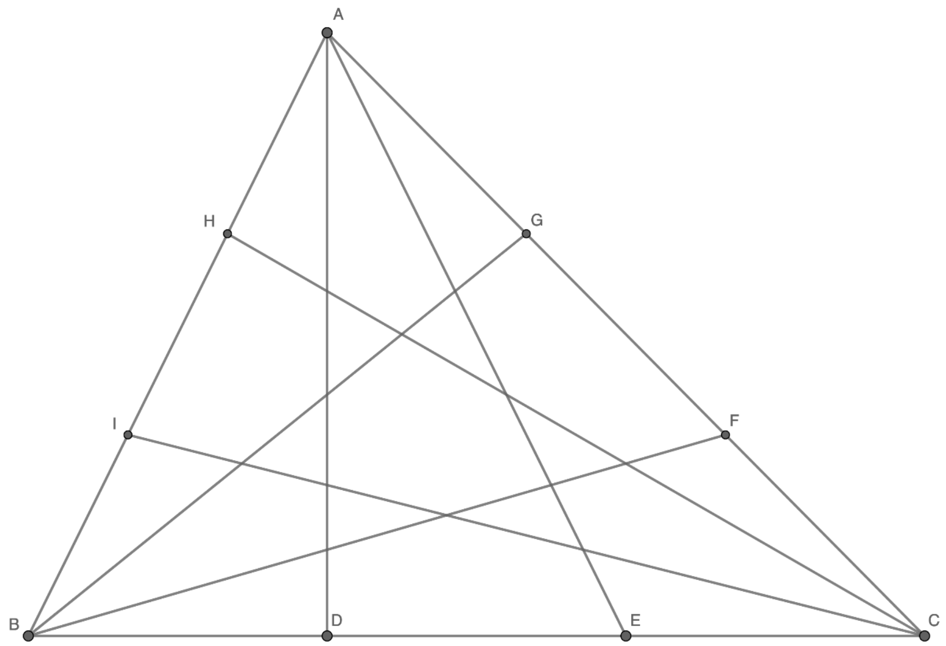
3. Marion's Theorem: Objectives

Consider an arbitrary $\triangle ABC$, with D and E trisecting BC, G and F trisecting AC, and H and I trisecting AB. When each vertex is joined to the trisection points on the opposite side, the area of the hexagon created inside the triangle is $\frac{1}{10}$ of $S_{\triangle ABC}$.



The report is separated into the following:

I. Marion's Theorem: Proof by Deductive Approach

II. Marion's Theorem: Areas of Other Polygons in the Triangle

III. Marion's Theorem: Isosceles and Equilateral Triangles

IV. Extension of Marion's Theorem: Quadrilaterals

8. Summary

In this report, we attempted to prove Marion's Theorem with a deductive approach. In Section 1, two methods were used to prove Marion's Theorem. In Sections 2 and 4, we aimed to find the correlation between the area of the hexagon and the area of the other regions in the triangle, as well as different properties of the octagon in the quadrilateral.

We also expanded on her discoveries by constructing variations of the original diagram, and attempted to prove the observations and special properties found, as shown in Section 3 and 4; the idea was extended to equilateral and isosceles triangles, and then to parallelograms, rhombuses, squares and rectangles. All the octagons formed in the middle had special angles and sides. We also managed to find the correlation between the number of equal parts a side of a parallelogram is divided into and the area of the octagon formed in the middle.

This project has connected the younger generation of mathematicians to the older generations. Although this all started by simply connecting lines together, we have achieved our goal at the end: "Celebrating Marion's Theorem: Correlation, Construction and Connection".

9. Reflection

Passion and Love: Developing an Interest in Geometry

From this maths project, we have challenged ourselves and expanded our knowledge beyond the textbook. The topic we chose for this project, Marion's Theorem and other related variations, have not only allowed us to put into practice the geometry skills we acquired during classes, but have also enriched our knowledge in geometry, and, most importantly, our interest and passion for mathematics. Our fascinating discoveries encouraged us to delve into geometry and we have truly experienced how intriguing the work of a mathematician is.

Through this project, we have taken our first glimpse into the beauty and elegance of geometry and mathematics.

Teamwork: Working Together in this Maths Project

Our project was a mix of individual work and collaboration. We divided work among ourselves but at the same time brainstormed solutions together. This was an effective approach to tackling problems and proofs in our project. Teamwork also motivated us in times when we faced obstacles and bottlenecks, and we helped each other out on different parts and had a lot of fun in the process, which greatly strengthened our bonds as friends.

COVID-19: Difficulties of Working Online

With the fifth wave of the pandemic and our project coming to an end, we look back on the beginning of the pandemic when we were forced to resort to online methods to collaborate and complete our project. Throughout this journey, we gained new knowledge on how to use online tools such as Geogebra to construct diagrams to better illustrate our ideas; we also discussed through Zoom meetings on long nights, which increased the efficiency of our work. Learning how to organise workload and communicate clearly was the main challenge for us. However, we soon developed a systematic approach to these problems by creating lists and weekly meetings to check on each others' progress. We were also very fortunate to have a great mentor teacher, who was always ready to give us feedback and pointers when we needed them.

The pandemic has not been easy for anyone, but we have adapted to the new normal. Our report reflects our hard work, flexibility and perseverance.

10. References

Art of Problem Solving, (2006). *AMC 10A Problems/Problem 17.* Retrieved from

https://artofproblemsolving.com/wiki/index.php/2006_AMC_10A_Problems/Problem_17